

LAGRANGIAN MECHANICS ON CENTERED SEMI-DIRECT PRODUCTS

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ABSTRACT. There exists two types of semi-direct products between a Lie group G and a vector space V . The left semi-direct product, $G \ltimes V$, can be constructed when G is equipped with a left action on V . Similarly, the right semi-direct product, $G \rtimes V$, can be constructed when G is equipped with a right action on V . In this paper, we will construct a new type of semi-direct product, $G \bowtie V$, which can be seen as the ‘sum’ of right and left semi-direct products. We then proceed to the parallel existing semi-direct product Euler-Poincaré theory. We find that the group multiplication, the Lie bracket, and the diamond operator can each be seen as a sum of the associated concepts in right and left semi-direct product theory. Finally, we conclude with a toy example and the group of 2-jets of diffeomorphisms above a fixed point. This final example has potential use in the creation of particle methods for problems on diffeomorphism groups.

1. INTRODUCTION

Let G be a Lie group and V be a vector space on which G acts by a left action. Given these ingredients, we may form the Lie group $G \ltimes V$, which is isomorphic to $G \times V$ as a set, but equipped with the composition

$$(g, v) \cdot_{\ltimes} (h, w) = (g \cdot h, g \cdot w + v) \quad , \quad \forall (g, v), (h, w) \in G \ltimes V.$$

A standard example of a system which evolves on a left semi-direct product is the heavy top, where $G = \text{SO}(3)$ and $V = \mathbb{R}^3$. In contrast, if G acts on V by a right action, we may form the right semi-direct product $G \rtimes V$ defined by the composition

$$(g, v) \cdot_{\rtimes} (h, w) = (g \cdot h, w + v \cdot h).$$

A standard example of such a system is a fluid with a vector-valued advected parameter [7]. In any case, it seems natural to surmise that the composition law

$$(1) \quad (g, v) \cdot_{\bowtie} (h, w) = (g \cdot h, g \cdot w + v \cdot h)$$

yields a new type of semi-direct product. The first result of this article is that (1) is a valid composition law in some circumstances, where the resulting group is dubbed a *centered semi-direct product*.

The second result is that the 2nd order Taylor expansions (or 2-jets) of diffeomorphisms over a fixed point is a centered semi-direct product. The main motivation behind understanding this example is to allow us to develop particle-based methods for fluid simulation, with applications ranging from medical imaging to simulation of complex fluids.

1.1. Background. The semi-direct product is a standard tool used in the construction of new Lie groups and plays an interesting role in geometric mechanics when the normal subgroup is interpreted as an advected parameter. A standard example is the modeling of the ‘heavy-top’, wherein the configuration space is the left semi-direct product $\text{SO}(3) \ltimes \mathbb{R}^3$ [7]. Another standard example is the modeling

of liquid crystals, in which we consider the right semi-direct product $\text{SDiff}(M) \ltimes V$. In this case, $\text{SDiff}(M)$ is the set of volume-preserving diffeomorphisms of a volume manifold M , and V is a vector space of Lie algebra valued one-forms on M upon which $\text{SDiff}(M)$ acts by pullback [5, 4]. Of course, the tangent bundle of a Lie group, TG , is isomorphic to a left semi-direct product $G \ltimes \mathfrak{g}$ by left-trivializing the group structure of TG . Additionally, TG is isomorphic to a right semi-direct product $G \ltimes \mathfrak{g}$ when the group structure of TG is right trivialized [1, see S5.3]. Thus, we see that this method of constructing groups can be found in a number of instances. In this article, we introduce a new type of semi-direct product which extends the existing semi-direct product theory.

A motivating example will be a desire to understand the Jet-groupoid of a manifold M . In particular, it is known that the jet-functor sends the diffeomorphism group to a groupoid known as the Jet groupoid [9, see S12]. As will be illustrated in S1.4, the isotropy groups of the jet groupoid for 2-jets have a group structure which can be written as a centered semi-direct product. A thorough understanding of the Jet groupoid can be useful for the creation of new particle-based methods wherein the particles carry jet data in addition to position and velocity data. The advantage of such a particle method is the possibility for a discrete form of Kelvin's circulation theorem [8]. Building such particle methods can be useful in scenarios in which one desires to work with the material representation of a fluid. This occurs in the image registration method known as 'Large Deformation Diffeomorphic Matching,' which is used in the field of computational anatomy [3]. Moreover, the potential energy associated with the advected parameters in complex fluids is often a function of certain gradients which require jet data in order to be advected by a diffeomorphism. Thus, keeping track of jet data may play a significant role in the construction of particle-based variational integrators.

1.2. Main Contributions. In this paper, we accomplish a sequence of goals, each building upon the previous. In particular:

- (1) In section 2, we define a new type of semi-direct product that we dub a *centered semi-direct product*.
- (2) In proposition 2.2, we derive the Lie algebra of a centered semi-direct product and its associated structures.
- (3) In section 3, we develop the Euler-Poincaré theory of centered semi-direct product in parallel with the existing theory of semi-direct product reduction [7].
- (4) In section 4, we describe the centered semi-direct product Euler-Poincaré equations for a few examples. We present one toy example before presenting the theory for an isotropy group of the 2-Jet groupoid.

Combined, these items allow for a computationally tractable algebraic understanding of 2-Jets and perhaps open the door to applications which were previously overlooked by geometric mechanicians.

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1.4. A motivating example. Let $\text{Diff}(M)$ denote the diffeomorphisms group of a manifold M . For a fixed $x \in M$ we may define the isotropy subgroup

$$\text{Iso}(x) = \{\varphi \in \text{Diff}(M) \mid \varphi(x) = x\}.$$

Let $\varphi \in \text{Iso}(x)$ and note that $T_x\varphi$ is a linear automorphism of the vector-space T_xM . In particular:

Proposition 1.1. *The functor “ T_x ” is a group homomorphism from $\text{Iso}(x)$ to $\text{GL}(T_xM)$.*

Proof. Clearly $\text{Iso}(x)$ and $\text{GL}(T_xM)$ are both Lie groups. Let $\varphi, \psi \in \text{Iso}(x)$. Then $T_x\varphi \circ T_x\psi = T_x(\varphi \circ \psi)$. \square

This observation has implications for computation for the following reason: By definition, $T_x\varphi$ approximates φ in a neighborhood of $x \in M$. Thus, if one desired to model a continuum with activity at x , then $T_x\varphi$ carries some of the crucial data to do this task. In particular, this is computationally tractable as the dimension of $\text{GL}(T_xM)$ is equal to $\dim(M)^2$. If $\dim(M) = n$ then $\text{GL}(T_xM) \equiv \text{GL}(n)$ is a Lie group of dimension n^2 .

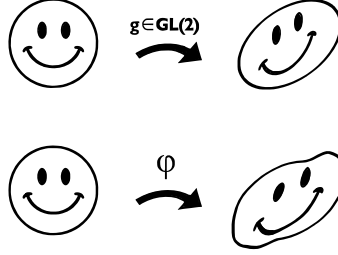


FIGURE 1. Depicted is a diffeomorphism with a trivial 2-jet (i.e. a linear transformation) and diffeomorphism with a nontrivial 2-jet.

However, the group $\text{GL}(n)$ only captures the linearization of a diffeomorphism. If we desire to capture some of the nonlinearity then we might consider looking into the second jet of these diffeomorphisms (see figure 1). We can do so by considering the functor TT_x . Let $\varphi \in \text{Iso}(x)$ so that $TT_x\varphi$ is a map from $T(T_xM)$ to $T(T_xM)$. However, T_xM is a vector-space so that $T(T_xM) \approx T_xM \times T_xM$. The second component represents the vertical component and the isomorphism between TT_xM and $T_xM \times T_xM$ is given by the vertical lift

$$v^\uparrow(v_1, v_2) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (v_1 + \epsilon v_2).$$

We can therefore represent $TT_x\varphi$ as $(T_x\varphi, A_\varphi)$ where $A_\varphi : T_xM \times T_xM \rightarrow T_xM$ is the symmetric $(1, 2)$ tensor

$$(2) \quad A_{ij}^k = \frac{\partial^2 \varphi^k}{\partial x_i \partial x_j}(x)$$

where φ^k is the k th component of φ . In other words, we have the 1-1 correspondence

$$TT_x\varphi \leftrightarrow (A_1, A_2)$$

where $A_1 = T_x\varphi$ and A_2 is given by (2). If we denote the set of symmetric T_xM -valued 2-tensors on T_xM by $\mathcal{S}_2^1(x)$, then this correspondence is given by a map

$$\Psi : \mathcal{J}^2|_x^x(\text{Diff}(M)) \rightarrow \text{GL}(T_xM) \times \mathcal{S}_2^1(x)$$

where $\mathcal{J}^2|_x^x(\text{Diff}(M))$ is the group of second order taylor expansions about x of diffeomorphisms which send x to itself (these are called 2-jets). This allows us to write the Lie group structure of $\mathcal{J}^2|_x^x(\text{Diff}(M))$ as a type of semi-direct product. In particular:

Proposition 1.2. *If we represent $TT_x\varphi$ and $TT_x\psi$ as (A_1, A_2) and (B_1, B_2) where $A_1 = T_x\varphi, B_1 = T_x\psi, A_2 = \frac{\partial^2\varphi^k}{\partial x^i\partial x^j}$, and $B_2 = \frac{\partial^2\psi^k}{\partial x^i\partial x^j}$, then $TT_x\varphi \circ TT_x\psi \equiv TT_x(\varphi \circ \psi)$ is given by the composition*

$$(A_1, A_2) \circ (B_1, B_2) = (A_1 \circ B_1, A_1 \circ B_2 + A_2 \circ (B_1 \times B_1)).$$

Proof. We find that

$$\frac{\partial}{\partial x_i}(\varphi^k \circ \psi) = \frac{\partial \varphi^k}{\partial x_l} \cdot \frac{\partial \psi^l}{\partial x_i} \circ \psi$$

and the second derivative is

$$\begin{aligned} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}(\varphi^k \circ \psi) &= \frac{\partial}{\partial x_j} \left(\frac{\partial \varphi^k}{\partial x_l} \cdot \frac{\partial \psi^l}{\partial x_i} \circ \psi \right) \\ &= \left(\frac{\partial^2 \varphi^k}{\partial x_l \partial x_m} \frac{\partial \psi^l}{\partial x_i} \frac{\partial \psi^m}{\partial x_j} + \frac{\partial \varphi^k}{\partial x_l} \frac{\partial^2 \psi^l}{\partial x_i \partial x_j} \right) \circ \psi \end{aligned}$$

Noting that $\psi(x) = x$ we can set

$$\begin{aligned} A_1 &= \left. \frac{\partial \varphi^k}{\partial x_l} \right|_x, & A_2 &= \left. \frac{\partial^2 \varphi^k}{\partial x_i \partial x_j} \right|_x \\ B_1 &= \left. \frac{\partial \psi^k}{\partial x_l} \right|_x, & B_2 &= \left. \frac{\partial^2 \psi^k}{\partial x_i \partial x_j} \right|_x \end{aligned}$$

and rewrite the equations in the form

$$\begin{aligned} \frac{\partial}{\partial x_i}(\varphi^k \circ \psi) &= A_1 \cdot B_1 \\ \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}(\varphi^k \circ \psi) &= A_1 \cdot B_2 + A_2 \circ (B_1 \times B_1) \end{aligned}$$

Therefore if we define the composition

$$(A_1, A_2) \cdot (B_1, B_2) := (A_1 \cdot B_1, A_1 \cdot B_2 + A_2 \circ (B_1 \times B_1))$$

on the manifold $\text{GL}(T_xM) \times \vee^2(T_xM; T_xM)$, then $\Psi : \mathcal{J}^2|_x^x(\text{Diff}(M)) \rightarrow \text{GL}(T_xM) \times \mathcal{S}_2^1$ is a Lie group isomorphism by construction. \square

We see that the composition law of Proposition 1.1 is of the form described in equation (1). In this paper, we will condense the composition law for 2-jets to the algebraic level and study (1) in the abstract Lie group setting. Of course, one would naturally like to consider diffeomorphisms which are not contained in $\text{Iso}(x)$. However, this extension brings us into the realm of Lie groupoid theory and will need to be addressed in future work.

2. A CENTERED SEMI-DIRECT PRODUCT THEORY

In this section, we will discover a new type of semi-direct product. We will outline the necessary ingredients for the construction of such a Lie group and we will derive the corresponding structures on the Lie algebra.

2.1. Preliminary material on Lie groups. Let G be a Lie group with identity $e \in G$ and Lie algebra \mathfrak{g} . In this subsection we will establish notation and recall relevant notions related to Lie groups and Lie algebras.

2.1.1. Group actions: Let V be a vector space. A *left action* of G on V is a smooth map $\rho_L : G \times V \rightarrow V$ for which:

$$\rho_L(e, v) = v \text{ and } \rho_L(g, \rho_L(h, v)) = \rho_L(gh, v) \quad , \quad \forall g, h \in G, \forall v \in V.$$

As using the symbol ' ρ_L ' can become cumbersome and since we will only need a one left Lie group action in a given context, we will opt to use the notation $g \cdot v := \rho_L(g, v)$. Finally, the *induced infinitesimal left action* of \mathfrak{g} on V is

$$\xi \cdot v := \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \exp(\epsilon \cdot \xi) \cdot v \quad , \quad \forall \xi \in \mathfrak{g}, v \in V.$$

Similarly, a *right action* of G on V is the smooth map $\rho_R : V \times G \rightarrow V$ for which:

$$\rho_R(v, e) = v \text{ and } \rho_R(\rho_L(v, g), h) = \rho_L(v, gh) \quad , \quad \forall g, h \in G, \forall v \in V.$$

Again, we will primarily use the notation $v \cdot g := \rho_R(v, g)$ for right actions. The *induced infinitesimal right action* of \mathfrak{g} on V is given by

$$v \cdot \xi = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} v \cdot \exp(\epsilon \cdot \xi) \quad , \quad \forall \xi \in \mathfrak{g}, v \in V$$

Lastly, we say that the left action and the right action *commute* if

$$(g \cdot v) \cdot h = g \cdot (v \cdot h)$$

for any $g, h \in G$ and $v \in V$.

2.1.2. Adjoint and coadjoint operators: In this section we will recall the “AD, Ad, ad”-notation used in [6]. For $g \in G$ we define the *inner automorphism* $\text{AD} : G \times G \rightarrow G$ as $\text{AD}(g, h) \equiv \text{AD}_g(h) = ghg^{-1}$. Differentiating AD with respect to the second argument along curves through the identity produces the *Adjoint representation* of G on \mathfrak{g} denoted $\text{Ad} : G \times \mathfrak{g} \rightarrow \mathfrak{g}$ and given by

$$\text{Ad}_g(\eta) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (\text{AD}_g(\exp(\epsilon\eta))) = g \cdot \eta \cdot g^{-1},$$

for $g \in G$ and $\eta \in \mathfrak{g}$. Differentiating Ad with respect to the first argument along curves through the identity produces the *adjoint operator* $\text{ad} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ given by

$$\text{ad}_\xi(\eta) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} (\text{Ad}_{\exp(\epsilon\xi)}(\eta)) = \xi \cdot \eta - \eta \cdot \xi.$$

The ad-map is an alternative notation for the Lie bracket of \mathfrak{g} in the sense that

$$\text{ad}(\xi, \eta) \equiv \text{ad}_\xi(\eta) \equiv [\xi, \eta].$$

For each $\xi \in \mathfrak{g}$ the map $\text{ad}_\xi : \mathfrak{g} \rightarrow \mathfrak{g}$ is linear and therefore has a formal dual $\text{ad}_\xi^* : \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ which we call the *coadjoint operator*. Explicitly, ad_ξ^* is defined by the relation

$$(3) \quad \langle \text{ad}_\xi^*(\mu), \eta \rangle = \langle \mu, \text{ad}_\xi(\eta) \rangle$$

for each $\eta \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$.

2.2. Centered semi-direct products. In this subsection, we will construct a semi-direct product which can be thought of as a ‘sum’ of a right semi-direct product and a left semi-direct product.

Proposition 2.1. *Let G be a Lie group which acts on a vector-space V via left and right group actions. Then, the product $G \times V$ with the composition law*

$$(4) \quad (g_1, v_1) \cdot (g_2, v_2) := (g_1 g_2, g_1 \cdot v_2 + v_1 \cdot g_2)$$

is a Lie group if and only if the left and right actions of G commute.

Proof. It is clear that $G \times V$ is a smooth manifold and that the composition law 4 is a smooth map. We must prove that this composition makes $G \times V$ a group.

- That the composition map (4) produces another element of $G \times V$ can be observed directly. Thus ‘closure’ is satisfied.
- The identity element is given by $(e, 0) \in G \times V$ where $e \in G$ is the identity of G .
- The inverse element of an arbitrary $(g, v) \in G \times V$ is $(g^{-1}, -g^{-1} v g^{-1})$ where g^{-1} is the inverse of $g \in G$.
- Given three elements of $G \times V$ we find

$$\begin{aligned} (g_1, v_1) \cdot ((g_2, v_2) \cdot (g_3, v_3)) &= (g_1, v_1) \cdot (g_2 g_3, g_2 \cdot v_3 + v_2 \cdot g_3) \\ &= (g_1 g_2 g_3, g_1 \cdot (g_2 \cdot v_3 + v_2 \cdot g_3) + v_1 \cdot (g_2 g_3)) \\ &= ((g_1 g_2) g_3, (g_1 g_2) \cdot v_3 + g_1 \cdot (v_2 \cdot g_3) + (v_1 \cdot g_2) \cdot g_3). \end{aligned}$$

By the commutativity of the group actions we may equate the above line with:

$$\begin{aligned} &= ((g_1 g_2) g_3, (g_1 g_2) \cdot v_3 + (g_1 \cdot v_2) \cdot g_3 + (v_1 \cdot g_2) \cdot g_3) \\ &= ((g_1 g_2) g_3, (g_1 g_2) \cdot v_3 + (g_1 \cdot v_2 + v_1 \cdot g_2) \cdot g_3) \\ &= ((g_1 g_2), g_1 \cdot v_2 + v_1 \cdot g_2) \cdot (g_3, v_3) \\ &= ((g_1, v_1) \cdot (g_2, v_2)) \cdot (g_3, v_3). \end{aligned}$$

Thus, the associative property is satisfied.

Moreover, all maps in sight including the inverse map are smooth. In conclusion we see that $G \times V$ with the composition 4 defines a Lie group. Moreover, if the left and right actions of G on V do *not* commute, then we can observe that associativity is violated. \square

Definition 2.1. *Given commuting left and right representations of a group G on a vector space V , the Lie group $G \times V$ with the composition (4) is denoted $G \bowtie V$ and called the centered semi-direct product of G and V .*

It is customary to denote the left semi-direct product using the symbol \ltimes and the right semi-direct product via the symbol \rtimes . We justify our use of the symbol \bowtie in that the concept of centered semi-direct product is merely a ‘sum’ of a left and a right semi-direct product. The formula $\bowtie = \ltimes + \rtimes$ can be used as a heuristic throughout the paper. In particular, this heuristic applies to the Lie algebra.

Proposition 2.2. *Let $G \bowtie V$ be a centered-semi direct product Lie group. The Lie algebra $\mathfrak{g} \bowtie V$ is given by the set $\mathfrak{g} \times V$ with the Lie bracket*

$$(5) \quad [(\xi_1, v_1), (\xi_2, v_2)]_{\bowtie} = ([\xi_1, \xi_2]_{\mathfrak{g}}, (\xi_1 \cdot v_2 + v_1 \cdot \xi_2) - (\xi_2 \cdot v_1 + v_2 \cdot \xi_1)),$$

for $\xi_1, \xi_2 \in \mathfrak{g}$, $v_1, v_2 \in V$.

Proof. Firstly, it is simple to verify that the tangent space at the identity, $(e, 0) \in G \times V$, is $\mathfrak{g} \times V$. To derive the Lie bracket, we will derive the the ad-map via the Ad and AD-maps. For $(g, v), (h, w) \in G \ltimes V$ we find

$$\begin{aligned} \text{AD}_{(g,h)}(h, w) &= (gh, v \cdot h + g \cdot w) \cdot (g^{-1}, -g^{-1} \cdot v \cdot g^{-1}) \\ &= (\text{AD}_g(h), v \cdot hg^{-1} + g \cdot w \cdot g^{-1} - \text{AD}_g(h) \cdot v \cdot g^{-1}). \end{aligned}$$

If we substitute (h, w) with the ϵ -dependent curve $(\exp(\epsilon \cdot \xi_2), \epsilon \cdot v_1)$ we can calculate the *adjoint operator*, $\text{Ad} : (G \ltimes V) \times (\mathfrak{g} \ltimes V) \rightarrow \mathfrak{g} \ltimes V$. Given by

$$\begin{aligned} \text{Ad}_{(g,v)}(\xi_2, v_2) &= \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \text{AD}_{(g,v)}(\exp(\epsilon \cdot \xi_1), \epsilon \cdot v_1) \\ &= (\text{Ad}_g(\xi_2); v \cdot \xi_2 g^{-1} + g \cdot v_2 \cdot g^{-1} - \text{Ad}_g(\xi_2) \cdot v \cdot g^{-1}). \end{aligned}$$

If we substitute (g, v) with the t -dependent curve $(\exp(t\xi_1), tv_2)$ we can differentiate with respect to t to produce the adjoint operator $\text{ad} : (\mathfrak{g} \ltimes V) \times (\mathfrak{g} \ltimes V) \rightarrow \mathfrak{g} \ltimes V$. Specifically, the adjoint operator is given by

$$\begin{aligned} \text{ad}_{(\xi_1, v_1)}(\xi_2, v_2) &= \left. \frac{d}{dt} \right|_{t=0} (\text{Ad}_{(\exp(t \cdot \xi_1), t \cdot v_1)}(\xi_2, v_2)) \\ &= \left. \frac{d}{dt} \right|_{t=0} (g\xi_2 g^{-1}, v \cdot \xi_2 g^{-1} - g\xi_2 g^{-1} \cdot v \cdot g^{-1} + g \cdot v_2 \cdot g^{-1}) \\ &= (\text{ad}_{\xi_1}(\xi_2), \xi_1 \cdot v_2 + v_1 \cdot \xi_2 - \xi_2 \cdot v_1 - v_2 \cdot \xi_1) \\ &= ([\xi_1, \xi_2]_{\mathfrak{g}}, (\xi_1 \cdot v_2 + v_1 \cdot \xi_2) - (\xi_2 \cdot v_1 + v_2 \cdot \xi_1)). \end{aligned}$$

Noting that the ad-map is merely an alternative notation for the Lie bracket completes the proof. \square

We complete this section by defining operations designed to express interaction terms between momenta in V in and momenta in G in mechanical systems.

Definition 2.2. The heart operator $\heartsuit : \mathfrak{g} \times V^* \rightarrow V^*$ is defined by

$$(6) \quad \langle \xi \heartsuit \alpha, v \rangle_V := \langle \alpha, \xi \cdot v - v \cdot \xi \rangle_V.$$

The diamond operator, $\diamond : V \times V^* \rightarrow \mathfrak{g}^*$, is defined as

$$(7) \quad \langle v \diamond \alpha, \xi \rangle_{\mathfrak{g}} := \langle \alpha, v \cdot \xi - \xi \cdot v \rangle_{\mathfrak{g}}.$$

The diamond operator can be seen as the sum of a diamond operator of a left semi-direct product and that of a right semi-direct product [7]. The heart operator will allow us to express how momenta in V impact motion in G , while the diamond operator will allow us to express the converse.

3. EULER-POINCARÉ THEORY

The Euler-Lagrange equations on a Lie group, \tilde{G} can be expressed by a vector field over $T\tilde{G}$. If the Lagrangian is \tilde{G} -invariant then the equations of motion are \tilde{G} -invariant as well and the evolution equations can be reduced. While the unreduced system evolves by the *Euler-Lagrange* equations on $T\tilde{G}$, the reduced dynamics evolve on the quotient $T\tilde{G}/\tilde{G}$. However, $T\tilde{G}/\tilde{G}$ is just an alternative description of the Lie algebra $\tilde{\mathfrak{g}}$ and so the reduced equations of motion can be described on $\tilde{\mathfrak{g}}$ where we call them the *Euler-Poincaré equations*. This reduction procedure is summarized by the commutative diagram:

$$\begin{array}{ccc}
T\tilde{G} & \xrightarrow{\text{flow by 'EL'}} & T\tilde{G} \\
\downarrow /_{\tilde{G}} & & \downarrow /_{\tilde{G}} \\
\tilde{\mathfrak{g}} & \xrightarrow{\text{flow by 'EP'}} & \tilde{\mathfrak{g}}
\end{array}$$

To be even more specific. A Lagrangian $L : T\tilde{G} \rightarrow \mathbb{R}$ is said to be *(right) \tilde{G} -invariant* if

$$L((\tilde{g}, \dot{\tilde{g}}) \cdot h) = L(\tilde{g}, \dot{\tilde{g}})$$

for all $h \in \tilde{G}$. If L is \tilde{G} -invariant, then L is uniquely specified by its restriction $\ell = L|_{\tilde{\mathfrak{g}}} : \tilde{\mathfrak{g}} \rightarrow \mathbb{R}$. The Euler-Poincaré theorem states the Euler Lagrange equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\tilde{g}}} \right) - \frac{\partial L}{\partial \tilde{g}} = 0$$

on $T\tilde{G}$ are equivalent the Euler-Poincaré equations and reconstruction formula

$$\frac{d}{dt} \left(\frac{\partial \ell}{\partial \tilde{\xi}} \right) = -\text{ad}_{\tilde{\xi}}^* \left(\frac{\partial \ell}{\partial \xi} \right), \quad \tilde{\xi} := \dot{\tilde{g}} \cdot \tilde{g}^{-1}.$$

A review of Euler-Poincaré reduction is given in [10, Ch 13] while a specialization to the case of semidirect products with advected parameters is described in [7]. In this section we will specialize the Euler-Poincaré theorem to the case of centered semi-direct products by setting $\tilde{G} = G \ltimes V$.

To begin let us compute how variations of curves in the group induce variations on the trivializations of the velocities to the Lie algebra. Studying such variations will allow us to transfer the variational principles on the group to variational principles on the Lie algebra.

Proposition 3.1. *Let $G \ltimes V$ be a centered semi-direct product and consider a curve $(g, v)(t) \in G \ltimes V$. Let $(\xi_g(t), \xi_v(t)) := (\dot{g}(t), \dot{v}(t)) \cdot (g(t), v(t))^{-1} \in \mathfrak{g} \ltimes V$ be the right trivialization of $(\dot{g}, \dot{v})(t)$. An arbitrary variation of $(g, v)(t)$ is given by*

$$(\delta g, \delta v)(t) = (\eta_g, \eta_v)(t) \cdot (g, v)(t) \in T_{(g, v)(t)}(G \ltimes V),$$

where $(\eta_g, \eta_v)(t) \in \mathfrak{g} \ltimes V$. Given such a variation, the induced variation on (ξ_g, ξ_v) is given by

$$\begin{aligned}
(8) \quad (\delta \xi_g, \delta \xi_v) &= (\dot{\eta}_g - \text{ad}_{\xi_g} \eta_g, \dot{\eta}_v + (\eta_g \xi_v + \eta_v \xi_g) - (\xi_g \eta_v + \xi_v \eta_g)) \\
&= \frac{d}{dt}(\eta_v, \eta_v) - [(\xi_g, \xi_v), (\eta_g, \eta_v)]_{\ltimes}.
\end{aligned}$$

Proof. For any Lie group, \tilde{G} , and any curve $\tilde{g}(t) \in \tilde{G}$, the variation of $\tilde{\xi}(t) := \dot{\tilde{g}}(t) \cdot \tilde{g}^{-1}(t)$ induced by the variation $\delta \tilde{g}(t) = \tilde{\eta}(t) \cdot \tilde{g}(t)$ is $\delta \tilde{\xi} = \dot{\tilde{\eta}} - [\tilde{\xi}, \tilde{\eta}]$. For matrix groups see [10, Theorem 13.5.3] and [2] for the general case. If we set $\tilde{G} = G \ltimes V$ and use the bracket derived in Proposition 2.2 then the theorem follows. \square

Now that we understand the relationship between variations of curves in $G \ltimes V$ and the induced variations in $\mathfrak{g} \ltimes V$ we can state the Euler-Poincaré theorem for centered semi-direct products.

Theorem 3.1. *Let $L : G \ltimes V \rightarrow \mathbb{R}$ be (right) $G \ltimes V$ -invariant, and let $\ell : \mathfrak{g} \ltimes V \rightarrow \mathbb{R}$ be its reduced Lagrangian. Let $(g, v)(t) \in G \ltimes V$ and denote the right trivialized velocity by $(\xi_g, \xi_v)(t) := (\dot{g}, \dot{v})(t) \cdot (g, v)(t)^{-1}$. Then the following statements are equivalent:*

(i) *Hamilton's principle holds. That is,*

$$(9) \quad \delta \int_{t_0}^{t_1} L(g(t), \dot{g}(t), v(t)) dt = 0$$

for variations of $(g, v)(t)$ with fixed endpoints.

(ii) *$(g, v)(t)$ satisfies the Euler-Lagrange equations for L .*

(iii) *The constrained variational principle*

$$(10) \quad \delta \int_{t_0}^{t_1} \ell(\xi_g(t), \xi_v(t)) dt = 0$$

holds on $\mathfrak{g} \times V$ for variations of the form

$$(11) \quad (\delta \xi_g, \delta \xi_v) = (\dot{\eta}_g - \text{ad}_{\xi_g} \eta_g, \dot{\eta}_v + \eta_g \xi_v - \xi_v \eta_g + \eta_v \xi_g - \xi_g \eta_v).$$

where $(\eta_g, \eta_v)(t)$ is an arbitrary curve in $\mathfrak{g} \bowtie V$ which vanishes at the endpoints.

(iv) *The Euler-Poincaré equations*

$$\begin{aligned} \frac{d}{dt} \left(\frac{\delta \ell}{\delta \xi_g} \right) + \text{ad}_{\xi_g}^* \left(\frac{\delta \ell}{\delta \xi_g} \right) + \xi_v \diamond \frac{\delta \ell}{\delta \xi_v} &= 0, \\ \frac{d}{dt} \left(\frac{\delta \ell}{\delta \xi_v} \right) + \xi_g \heartsuit \frac{\delta \ell}{\delta \xi_v} &= 0 \end{aligned}$$

hold on $\mathfrak{g} \bowtie V$.

Proof. The equivalence (i) and (ii) holds for any configuration manifold and so, in particular it holds in this case.

Next we show the equivalence (iii) and (iv). We compute the variations of the action integral to be

$$\begin{aligned} \delta \int_{t_0}^{t_1} \ell(\xi_g(t), \xi_v(t)) dt &= \int_{t_0}^{t_1} \left\langle \frac{\delta \ell}{\delta \xi_g}, \delta \xi_g \right\rangle + \left\langle \frac{\delta \ell}{\delta \xi_v}, \delta \xi_v \right\rangle dt \\ &= \int_{t_0}^{t_1} \left\langle \frac{\delta \ell}{\delta \xi_g}, \dot{\eta}_g - \text{ad}_{\xi_g} \eta_g \right\rangle + \left\langle \frac{\delta \ell}{\delta \xi_v}, \dot{\eta}_v + \eta_g \xi_v - \xi_v \eta_g + \eta_v \xi_g - \xi_g \eta_v \right\rangle dt \end{aligned}$$

and applying integration by parts and equation (3) we find

$$\begin{aligned} &= \int_{t_0}^{t_1} \left\langle -\frac{d}{dt} \left(\frac{\delta \ell}{\delta \xi_g} \right) - \text{ad}_{\xi_g}^* \left(\frac{\delta \ell}{\delta \xi_g} \right), \eta_g \right\rangle + \left\langle -\frac{d}{dt} \frac{\partial \ell}{\partial \xi_v}, \eta_v \right\rangle \\ &\quad + \left\langle \frac{\delta \ell}{\delta \xi_v}, \eta_g \xi_v - \xi_v \eta_g \right\rangle + \left\langle \frac{\delta \ell}{\delta \xi_v}, \eta_v \xi_g - \xi_g \eta_v \right\rangle dt \\ &\quad + \left\langle \frac{\partial \ell}{\partial \xi_g}, \eta_g \right\rangle \Big|_{t_0}^{t_1} + \left\langle \frac{\delta \ell}{\delta \xi_v}, \eta_v \right\rangle \Big|_{t_0}^{t_1} \\ &= \int_{t_0}^{t_1} \left\langle -\frac{d}{dt} \left(\frac{\delta \ell}{\delta \xi_g} \right) - \text{ad}_{\xi_g}^* \left(\frac{\delta \ell}{\delta \xi_g} \right) - \left(\xi_v \diamond \frac{\delta \ell}{\delta \xi_v} \right), \eta_g \right\rangle \\ &\quad + \left\langle -\frac{d}{dt} \left(\frac{\delta \ell}{\delta \xi_v} \right) - \xi_g \heartsuit \frac{\delta \ell}{\delta \xi_v}, \eta_v \right\rangle dt. \end{aligned}$$

By noting that $(\eta_g, \eta_v)(t)$ is arbitrary on the interior of the integration domain, the result follows.

Finally, we show that (i) and (iii) are equivalent. The G -invariance of L implies that the integrands in (9) and (10) are equal. However, by Proposition 3.1 all the variations of $(g, v)(t)$ with fixed endpoints induce, and are induced by, variations

$(\delta\xi_g, \delta\xi_v)(t) \in \mathfrak{g} \ltimes V$ of the form given in equation (11). Conversely if (i) holds with respect to arbitrary variations $(\delta g, \delta v)$, we define

$$(\eta_g, \eta_v)(t) = (\delta g, \delta v) \cdot (g, v)^{-1},$$

to produce the variation of (ξ_g, ξ_v) given in equation (11). \square

Remark 3.1. *There is a left invariant version of theorem (3.1) in which $(\xi_g, \xi_v) := (g, v)^{-1} \cdot (\dot{g}, \dot{v})$ and L is left $G \ltimes V$ -invariant. In this case the Euler-Poincaré equations take the form*

$$\begin{aligned} \frac{d}{dt} \left(\frac{\delta \ell}{\partial \xi_g} \right) - \text{ad}_{\xi_g}^* \left(\frac{\delta \ell}{\partial \xi_g} \right) - \xi_v \diamond \frac{\delta \ell}{\partial \xi_v} &= 0, \\ \frac{d}{dt} \left(\frac{\delta \ell}{\partial \xi_v} \right) - \xi_g \heartsuit \frac{\delta \ell}{\partial \xi_v} &= 0. \end{aligned}$$

Remark 3.2. *There is a version of semi-direct product mechanics wherein the vector-space V is a set of advected parameters as in [7]. In this case we impose the holonomic constraint*

$$\dot{v} = \dot{g} \cdot v + v \cdot \dot{g}$$

and the set of admissible variations in $\mathfrak{g} \ltimes V$ become

$$\delta \xi_g = \dot{\eta}_g - [\xi_g, \eta_g] \quad , \quad \delta v = \eta_g \cdot v + v \cdot \eta_g.$$

If we do this, the \heartsuit -term is removed and $\frac{\delta \ell}{\delta v}$ equation is replaced with a holonomic constraint. In particular we find that

$$\begin{aligned} \frac{d}{dt} \left(\frac{\delta \ell}{\partial \xi_g} \right) \pm \text{ad}_{\xi_g}^* \left(\frac{\delta \ell}{\partial \xi_g} \right) \pm \xi_v \diamond \frac{\delta \ell}{\partial \xi_v} &= 0 \\ \frac{dv}{dt} &= \xi_g \cdot v + v \cdot \xi_g. \end{aligned}$$

where we use a plus sign for right trivialization and a minus sign for left trivialization.

4. EXAMPLES

In this section we will present two examples of Euler-Poincaré equations on centered semidirect products. This first is a toy example designed to illustrate how computations of the diamond and heart operators can be done in practice. The second example is designed to 2-Jets as described in subsection 1.4.

4.1. A toy example. Consider the group $GL(n)$ and let $\text{Mat}(n)$ denote the vector space of $n \times n$ real matrices. Noting that $GL(n)$ acts on $\text{Mat}(n)$ by left and right multiplication, we can define the composition law on the Lie group $GL(n) \ltimes \text{Mat}(n)$ by:

$$(A, v) \cdot (B, w) = (AB, Aw + vB).$$

Moreover, we can identify $\mathfrak{gl}^*(n)$ with $\mathfrak{gl}(n)$ and $\text{Mat}(n)^*$ with $\text{Mat}(n)$ by the matrix trace pairing $\langle A, B \rangle = \text{Tr}(A^T B)$. This allows us to calculate the heart operator $\heartsuit : \mathfrak{gl}(n) \times \text{Mat}(n)^* \rightarrow \text{Mat}(n)$ as

$$\begin{aligned}
\langle A \heartsuit w, v \rangle &= \langle w, A \cdot v - v \cdot A \rangle \\
&= \text{trace}(w^T(A \cdot v - v \cdot A)) \\
&= \text{trace}(w^T \cdot (A \cdot v) - w^T(v \cdot A)) \\
&= \text{trace}((w^T \cdot A)v - (A \cdot w^T) \cdot v) \\
&= \text{trace}((w^T \cdot A - A \cdot w^T) \cdot v) \\
&= \text{trace}((A^T w - w \cdot A^T)^T \cdot v) \\
&= \langle A^T w - w A^T, v \rangle
\end{aligned}$$

Therefore,

$$A \heartsuit w = A^T w - w A^T.$$

By a similar calculation, diamond operator is found to be

$$v \diamond w = v^T w - w v^T,$$

and the coadjoint action on $\text{GL}(n)$ is given by

$$\text{ad}_A^*(\alpha_A) = A^T \cdot \alpha_A - \alpha_A \cdot A^T.$$

Now, we have all the ingredients to write the Euler-Poincaré equations. Given a reduced Lagrangian $\ell : \mathfrak{gl}(n) \ltimes \text{Mat}(n) \rightarrow \mathbb{R}$ we may denote the reduced momenta by

$$\mu = \frac{\delta \ell}{\delta \xi}, \quad \gamma = \frac{\delta \ell}{\delta v}.$$

where $(\xi, v) \in \mathfrak{gl}(n) \ltimes \text{Mat}(n)$. The Euler-Poincaré equations can be written as

$$\begin{aligned}
\dot{\mu} &= (\xi^T \mu - \mu \xi^T) + v^T \gamma - \gamma v^T \\
\dot{\gamma} &= \xi^T \gamma - \gamma \xi^T.
\end{aligned}$$

4.2. An isotropy group of a 2-Jet groupoid. In proposition 1.1 we illustrated how the set of 2-jets of diffeomorphisms of the stabilizer group of a point $x \in M$ is identifiable with a centered semidirect product. In particular, if $\dim(M) = n$ we can consider the group $\text{GL}(n) \ltimes \mathcal{S}_2^1$, where \mathcal{S}_2^1 is the set of $(1, 2)$ -tensors which are symmetric in the covariant part. For the moment we shall consider the larger space of all $(1, 2)$ -tensors denoted \mathcal{T}_2^1 . If we let $\mathbf{e}_1, \dots, \mathbf{e}_n \in \mathbb{R}^n$ be a basis with dual basis $\mathbf{e}^1, \dots, \mathbf{e}^n \in (\mathbb{R}^n)^*$ we can write an arbitrary element of \mathcal{T}_2^1 as

$$T = T_{jk}^i \mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k.$$

The left action of $\text{GL}(n)$ on \mathcal{T}_2^1 is

$$g \cdot T := T_{jk}^i (g \cdot \mathbf{e}_i) \otimes \mathbf{e}^j \otimes \mathbf{e}^k \equiv T_{jk}^i g_i^l \mathbf{e}_l \otimes \mathbf{e}^j \otimes \mathbf{e}^k$$

while the right action is

$$T \cdot g := T_{jk}^i \mathbf{e}_i \otimes (g^T \cdot \mathbf{e}^j) \otimes (g^T \cdot \mathbf{e}^k).$$

Clearly these actions commute, and so we may form the centered semidirect product Lie group $\text{GL}(n) \ltimes \mathcal{T}_2^1$.

Let us now focus on the Lie algebra. The Lie algebra $\mathfrak{gl}(n)$ is equivalent to \mathcal{T}_1^1 and the Lie bracket is then given in the bases $\mathbf{e}_i \otimes \mathbf{e}^j$ by

$$[\xi, \eta] = (\xi_k^i \eta_j^k - \eta_k^i \xi_j^k) \mathbf{e}_i \otimes \mathbf{e}^j,$$

where $\xi = \xi_j^i \mathbf{e}_i \otimes \mathbf{e}^j$ and $\eta = \eta_j^i \mathbf{e}_i \otimes \mathbf{e}^j$. We can use the dual basis $\mathbf{e}^i \otimes \mathbf{e}_j$ to see that the coadjoint action of ξ on $\mu = \mu_i^j \mathbf{e}^i \otimes \mathbf{e}_j$ is given by

$$\text{ad}_\xi^* \mu = (\mu_k^j \xi_i^k - \mu_i^k \xi_k^j) \mathbf{e}^i \otimes \mathbf{e}_j.$$

By differentiation we see that the infinitesimal left and right actions of $\mathfrak{gl}(n)$ on \mathcal{T}_2^1 are given by

$$\begin{aligned}\xi \cdot T &= T_{jk}^i \xi_i^l \mathbf{e}_l \otimes \mathbf{e}^j \otimes \mathbf{e}^k \\ T \cdot \xi &= T_{lk}^i \left[\mathbf{e}_i \otimes (\xi_l^j \cdot \mathbf{e}^l) \otimes \mathbf{e}^k + \mathbf{e}_i \otimes \mathbf{e}^j \otimes (\xi_l^k \cdot \mathbf{e}^l) \right] \\ &= (T_{lk}^i \xi_j^l + T_{jl}^i \xi_k^l) \mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k.\end{aligned}$$

If we choose an arbitrary element $\alpha \in (\mathcal{T}_2^1)^* \equiv \mathcal{T}_1^2$ given by

$$\alpha = \alpha_i^{jk} \mathbf{e}^i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

we find that

$$\begin{aligned}\langle \alpha, \xi \cdot T \rangle &= (\alpha_l^{jk} \xi_i^l) T_{jk}^i = (\alpha_i^{lk} T_{lk}^j) \xi_j^i \\ \langle \alpha, T \cdot \xi \rangle &= (\alpha_i^{lk} \xi_l^j + \alpha_i^{jl} \xi_l^k) T_{jk}^i = (\alpha_l^{jk} T_{ik}^l + \alpha_l^{kj} T_{ki}^l) \xi_j^i.\end{aligned}$$

Therefore the heart operator is given by

$$\xi \heartsuit \alpha = (\xi_i^l \alpha_l^{jk} - \alpha_i^{lk} \xi_l^j - \alpha_i^{jl} \xi_l^k) \mathbf{e}^i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

and the diamond operator is

$$\alpha \diamond T = (\alpha_l^{jk} T_{ik}^l + \alpha_l^{kj} T_{ki}^l - \alpha_i^{lk} T_{lk}^j) \mathbf{e}^i \otimes \mathbf{e}_j.$$

Given a reduced Lagrangian $\ell : \mathfrak{gl}(n) \ltimes \mathcal{T}_2^1 \rightarrow \mathbb{R}$ we can denote $\mu = \frac{\delta \ell}{\delta \xi}$ and $\gamma = \frac{\delta \ell}{\delta T}$. In terms of the basis $\mathbf{e}^i \otimes \mathbf{e}_j$ and $\mathbf{e}_i \otimes \mathbf{e}^j \otimes \mathbf{e}^k$ we may write the (right) Euler-Poincaré equations as:

$$\begin{aligned}\dot{\mu}_i^j &= \alpha_i^{lk} T_{lk}^j + \mu_k^j \xi_i^k - \mu_i^k \xi_k^j - \alpha_l^{jk} T_{ik}^l - \alpha_l^{kj} T_{ki}^l \\ \dot{T}_{jk}^i &= \xi_i^l \alpha_l^{jk} - \alpha_i^{lk} \xi_l^j - \alpha_i^{jl} \xi_l^k.\end{aligned}$$

By restricting \mathcal{T}_2^1 to the subspace \mathcal{S}_2^1 , we can obtain a Lie group which models 2-jets of diffeomorphisms as demonstrated in proposition 1.2. This example provides a first step towards the creation of higher-order, spatially accurate particle methods [8, section 4.2]. Moreover, the data of 2-jets is necessary for the advection of quantities seen in complex fluids in which the advected parameters depend on gradients of the flow [4, 5]. Therefore, the structures described here may prove useful in the construction of particle-based integrators for complex fluids as well.

5. CONCLUSION

In this paper, we have presented a variant of traditional semi-direct products, dubbed centered semi-direct products, and we have illustrated the associated Euler-Poincaré theory. The diamond operator, the group multiplication, and the Lie bracket can all be seen as sums of the associated concepts for left and right semi-direct products. As a result, the Euler-Poincaré theory associated with centered semi-direct products can also be seen as a sum of the left and right invariant Euler-Poincaré theories for semi-direct products. Presently, many of these constructions remain fairly theoretical. However, an isotropy group of the groupoid of 2-jets of diffeomorphisms of a manifold can be seen as a centered semi-direct product. This has potential applications in simulation of complex fluids. We hope this paper provides a stepping stone towards realizing this application.

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